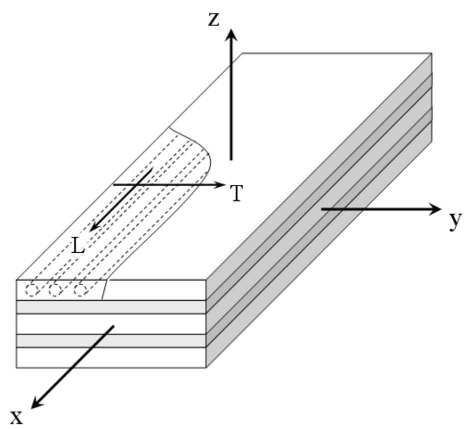
Competence Centers for Excellent Technologies

Analytical Stress-Strain analysis of the laminates with orthotropic (isotropic) layers using Classical Laminate Theory

+ Comparison with FEA



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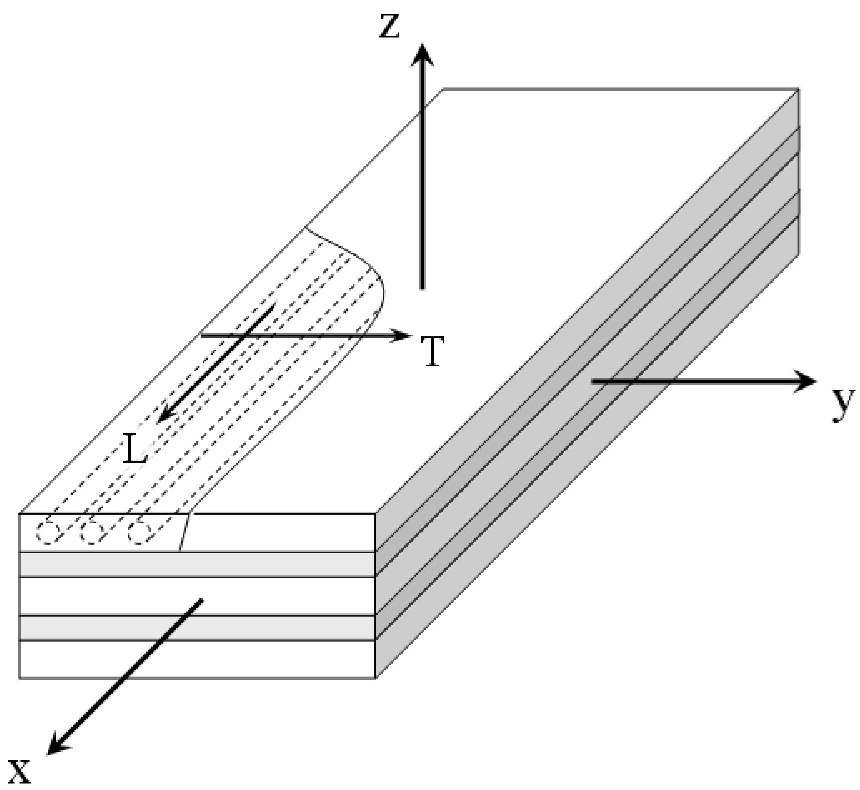
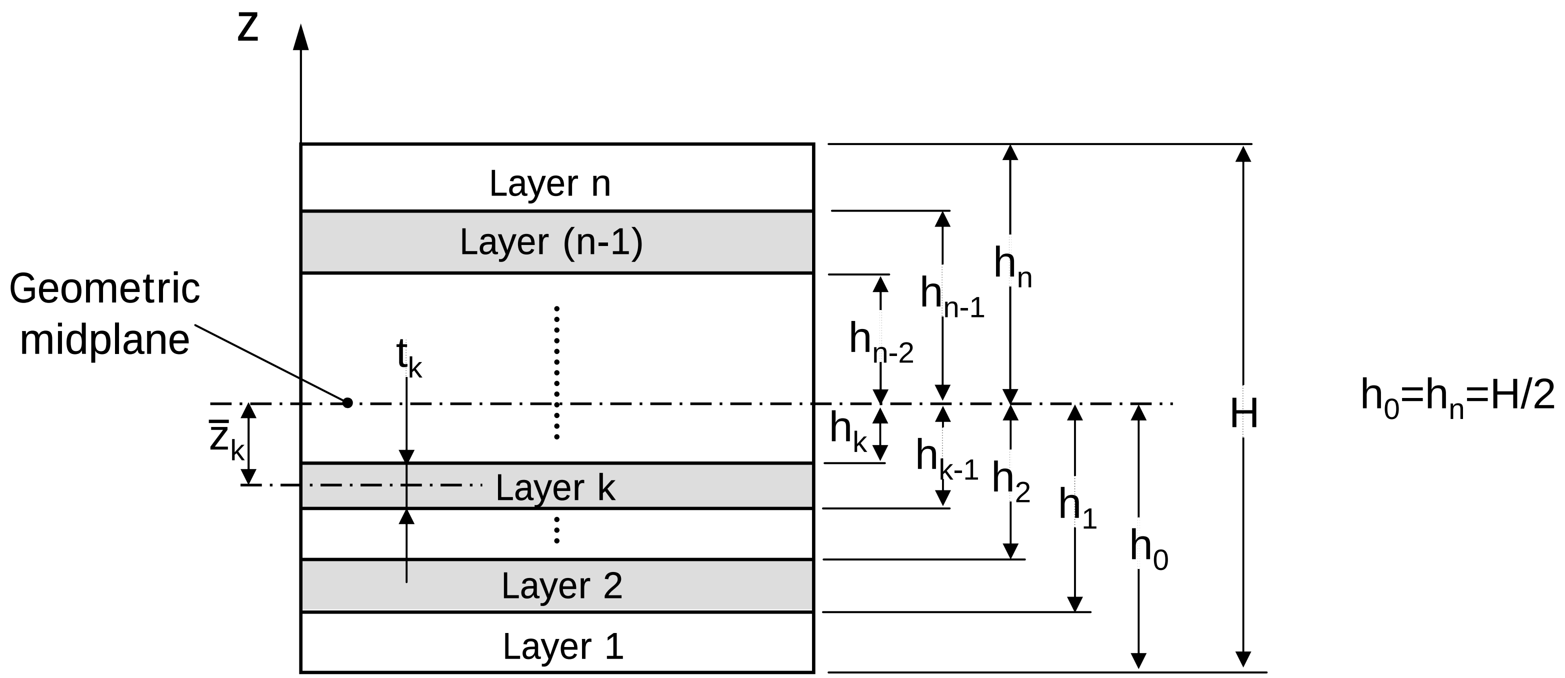
Motivation

- Fast insight on the laminate behaviour without FEA
- Creation of the interactive Mathematica applet for the fast laminate design (tailoring)



Laminate theory

- Main idea of the Laminate Theory
  - Set up the global laminate stiffeness matrix („Hookes´ law“).
  - Solve the global laminate behaviour.
  - Return back to single layers and solve the desired quantities.
- Scheme of the laminate and used notation



Laminate theory

- Properties of the single layer

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} = \begin{bmatrix} 1/E_L & -\nu_{LT}/E_L & 0 \\ -\nu_{TL}/E_T & 1/E_T & 0 \\ 0 & 0 & 1/G_{LT} \end{bmatrix}$$

Compliance matrix in the material CS

$$\mathbf{T} = \begin{bmatrix} \cos^2\phi & \sin^2\phi & -\sin\phi\cos\phi \\ \sin^2\phi & \cos^2\phi & \sin\phi\cos\phi \\ 2\sin\phi\cos\phi & -2\sin\phi\cos\phi & \cos^2\phi - \sin^2\phi \end{bmatrix}$$

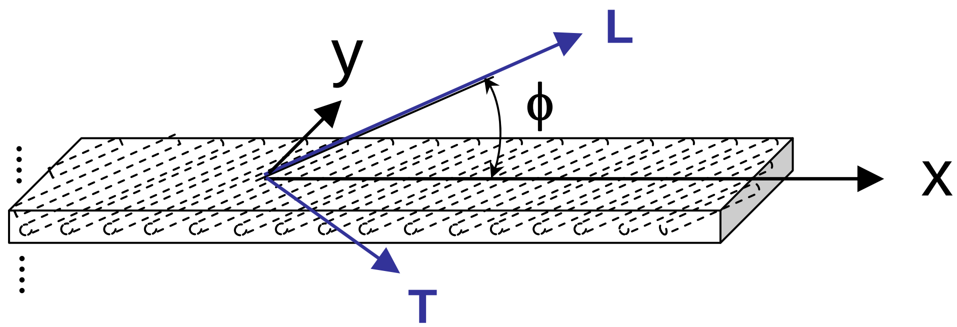
Transformation matrix – rotation of material compliance matrix into xy CS

$$\overline{\mathbf{C}} = \mathbf{T}\cdot\mathbf{C}\cdot\mathbf{T}^T$$

Compliance matrix in CS xy

$$\overline{\mathbf{S}} = \overline{\mathbf{C}}^{-1}$$

Stiffeness matrix in CS xy



$$\boldsymbol{\varepsilon}_{el} = \overline{\mathbf{C}}\cdot\boldsymbol{\sigma}$$

Laminate theory

- Properties of the single layer

Rotation of the CTEs vector

$$\mathbf{C}_{\alpha} = \begin{bmatrix} \alpha_L \\ \alpha_T \\ 0 \end{bmatrix}$$

Vector of CTEs in material CS

$$\overline{\mathbf{C}}_{\alpha} = \mathbf{T} \cdot \mathbf{C}_{\alpha} = \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}$$

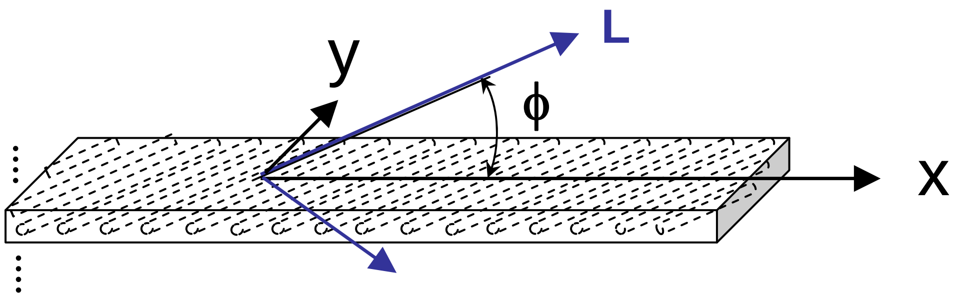
Vector of CTEs in CS xy

$$\overline{\mathbf{S}}_{\alpha} = \overline{\mathbf{S}} \cdot \overline{\mathbf{C}}_{\alpha}$$

„Stiffeness“ temperature vector in CS xy

$$\boldsymbol{\sigma} = \overline{\mathbf{S}} \cdot \boldsymbol{\varepsilon} - \overline{\mathbf{S}}_{\alpha} \cdot \Delta T$$

Hooke´s law of one layer



$$\boldsymbol{\varepsilon}_{th} = \overline{\mathbf{C}}_{\alpha} \cdot \Delta T$$

$$\boldsymbol{\varepsilon}_{tot} = \overline{\mathbf{C}} \cdot \boldsymbol{\sigma} + \overline{\mathbf{C}}_{\alpha} \cdot \Delta T$$

Laminate theory

- Calculation of the whole laminate

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-H/2}^{H/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} dz \qquad \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-H/2}^{H/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} z\, dz$$

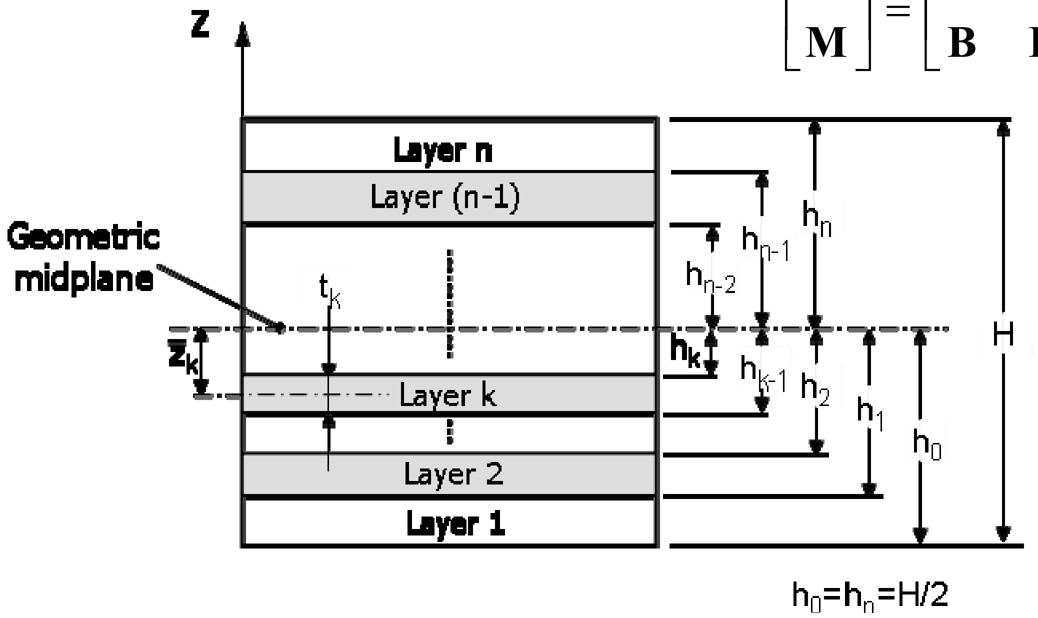
Stress and moment equilibrium over laminate height

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{21} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \cdot \left[\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_x^0 \\ \varepsilon_{xy}^0 \end{bmatrix} + z \cdot \begin{bmatrix} \kappa_x^0 \\ \kappa_x^0 \\ \kappa_{xy}^0 \end{bmatrix} - \begin{bmatrix} \alpha_x \\ \alpha_x \\ \alpha_{xy} \end{bmatrix} \cdot \Delta T \right]$$

Hooke´s law of one layer

Extension Bending Temperature

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa}^0 \end{bmatrix} - \begin{bmatrix} \mathbf{N}_{th} \\ \mathbf{M}_{th} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa}^0 \end{bmatrix} - \begin{bmatrix} \mathbf{N}_{th} \\ \mathbf{M}_{th} \end{bmatrix}$$



$$A_{ij} = \sum_{k=1}^{n} \left[\bar{S}_{ij} \right]_k \left(h_k - h_{k-1} \right)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left[\bar{S}_{ij} \right]_k \left(h_k^2 - h_{k-1}^2 \right)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left[\bar{S}_{ij} \right]_k \left(h_k^3 - h_{k-1}^3 \right)$$

Laminate theory

- Stiffeness matrix of the whole laminate

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa}^0 \end{bmatrix} - \begin{bmatrix} \mathbf{N}_{th} \\ \mathbf{M}_{th} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa}^0 \end{bmatrix} - \begin{bmatrix} \mathbf{N}_{th} \\ \mathbf{M}_{th} \end{bmatrix}$$

Thermal Stress and Moment resultants

External Stress and Moment resultants

Stiffeness matrix of the laminate

Midplane deformations

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\ A_{21} & A_{22} & A_{23} & B_{21} & B_{22} & B_{23} \\ A_{31} & A_{32} & A_{33} & B_{31} & B_{32} & B_{33} \\ B_{11} & B_{12} & B_{13} & D_{11} & D_{12} & D_{13} \\ B_{21} & B_{22} & B_{23} & D_{21} & D_{22} & D_{23} \\ B_{31} & B_{32} & B_{33} & D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x^0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{bmatrix} - \begin{bmatrix} N_{x,th} \\ N_{y,th} \\ N_{xy,th} \\ M_{x,th} \\ M_{y,th} \\ M_{xy,th} \end{bmatrix}$$

„Hooke´s law“ for the whole laminate

known (green): N, M, A, B, D matrix, N_{th}, M_{th}

unknown (red): ε^0, κ^0

$$\mathbf{N_{th}} = \sum_{k=1}^{n} \overline{\mathbf{S}}_k \cdot \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_k \Delta T \cdot t_k \qquad \mathbf{M_{th}} = \sum_{k=1}^{n} \overline{\mathbf{S}}_k \cdot \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_k \Delta T \cdot t_k \cdot \overline{z}_k$$

Laminate theory

- Calculation of strain and stresses

$$\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa}^0 \end{bmatrix} = \mathbf{K}^{-1}\left[\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} + \begin{bmatrix} \mathbf{N}_{th} \\ \mathbf{M}_{th} \end{bmatrix}\right]$$

Deformation and curvatures of the midplane

Strains over laminate height

$$\begin{bmatrix} \varepsilon_{x,th} \\ \varepsilon_{y,th} \\ \varepsilon_{xy,th} \end{bmatrix}_k = \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_k \cdot \Delta T$$

$$\begin{bmatrix} \varepsilon_{x,tot} \\ \varepsilon_{y,tot} \\ \varepsilon_{xy,tot} \end{bmatrix} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \end{bmatrix} + z \cdot \begin{bmatrix} \kappa_x^0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{bmatrix}$$

$$\varepsilon_{el} = \varepsilon_{tot} - \varepsilon_{th}$$

Stresses over laminate height

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_z = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{21} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix}_z \cdot \left[\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_x^0 \\ \varepsilon_{xy}^0 \end{bmatrix} + z \cdot \begin{bmatrix} \kappa_x^0 \\ \kappa_x^0 \\ \kappa_{xy}^0 \end{bmatrix} - \begin{bmatrix} \alpha_x \\ \alpha_x \\ \alpha_{xy} \end{bmatrix}_z \cdot \Delta T\right] = \left[\bar{S}\right]_z \left[\boldsymbol{\varepsilon}_{\mathbf{el}}\right]_z$$

Laminate theory

- Apparent properties of the whole laminate

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix}; \qquad \mathbf{K}^{-1} = \begin{bmatrix} \overline{\mathbf{A}} & \overline{\mathbf{B}} \\ \overline{\mathbf{B}} & \overline{\mathbf{D}} \end{bmatrix}$$

- Apparent E-modul, Poissons´ ratios ν, G-modul

$$E_{x,app} = \frac{1}{H \cdot \overline{A}_{11}}$$

$$E_{y,app} = \frac{1}{H \cdot \overline{A}_{22}}$$

$$\nu_{xy,app} = \frac{A_{12}}{A_{22}} = -\frac{\overline{A}_{12}}{\overline{A}_{11}}$$

$$\nu_{yx,app} = \frac{A_{12}}{A_{11}} = -\frac{\overline{A}_{12}}{\overline{A}_{22}}$$

$$G_{xy,app} = G_{yx,app} = A_{66} = \frac{1}{H \cdot \overline{A}_{66}}$$

H – total height of the laminate

- Apparent CTEs

$$\begin{bmatrix} \alpha_x^0 \\ \alpha_x^0 \\ \alpha_{xy}^0 \end{bmatrix} = \mathbf{K}^{-1} \cdot \begin{bmatrix} N_{x,th} \\ N_{x,th} \\ N_{xy,th} \end{bmatrix} \cdot \frac{1}{\Delta T}$$

Laminate theory

- Deformation of the laminate (plate curvatures)

$$-\frac{\partial^2 w}{\partial x^2} = \kappa_x \Rightarrow w(x) = \int\int -\kappa_x \, dx \, dx = -\kappa_x \cdot \frac{x^2}{2}$$

deformation along x axis

$$-\frac{\partial^2 w}{\partial x^2} = \kappa_y \Rightarrow w(y) = \int\int -\kappa_y \, dy \, dy = -\kappa_y \cdot \frac{y^2}{2}$$

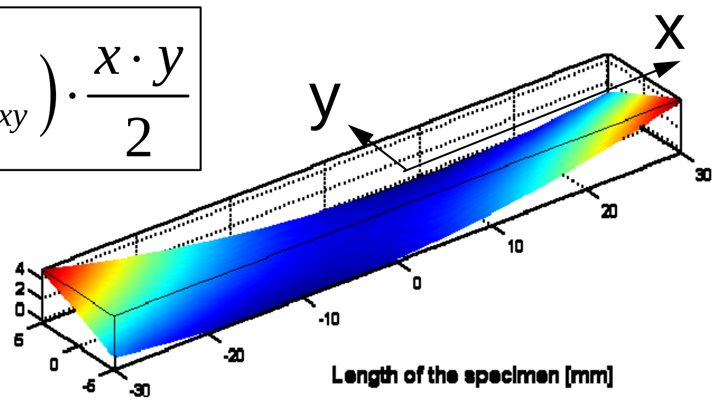
deformation along y axis

$$-2\frac{\partial^2 w}{\partial x \partial y} = \kappa_{xy} \Rightarrow w(x,y) = \int\int -\kappa_{xy} \, dy \, dx = -\kappa_{xy} \cdot \frac{x \cdot y}{2}$$

twisting deformation

$$w_{tot}(x,y) = w(x) + w(y) + w(x,y)$$

$$\boxed{w_{tot}(x,y) = (-\kappa_x) \cdot \frac{x^2}{2} + (-\kappa_y) \cdot \frac{y^2}{2} + (-\kappa_{xy}) \cdot \frac{x \cdot y}{2}}$$

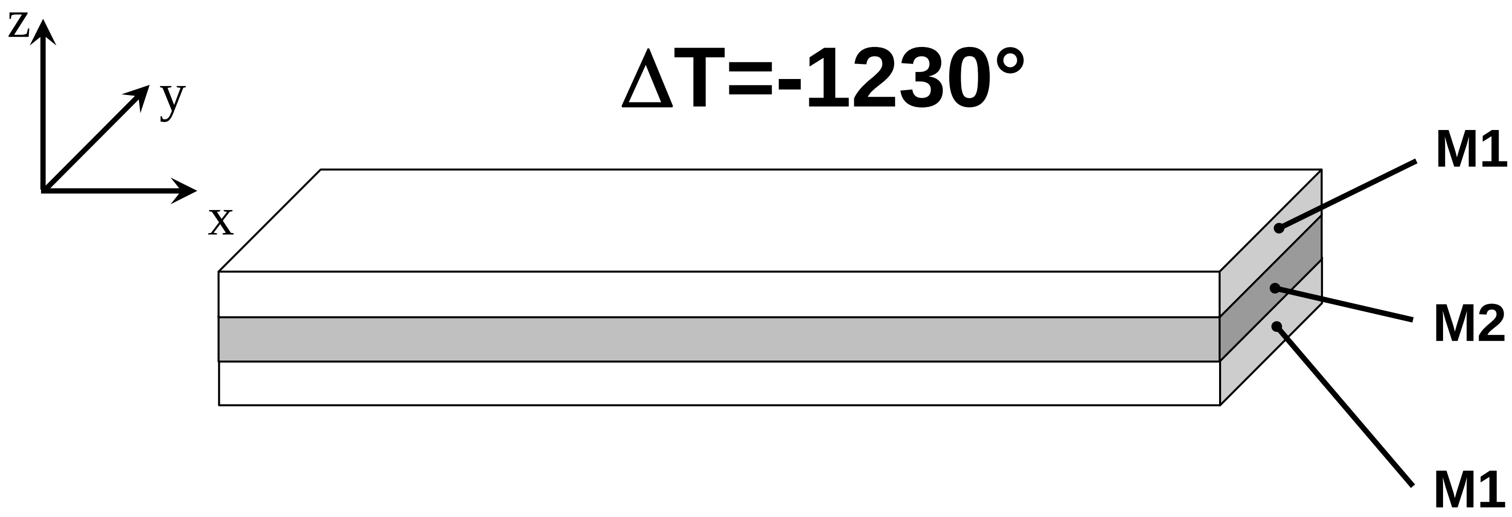


Examples

(for pure thermal loading)

Examples

1) Symmetric laminate with isotropic layers:



TH_{M1}=1mm

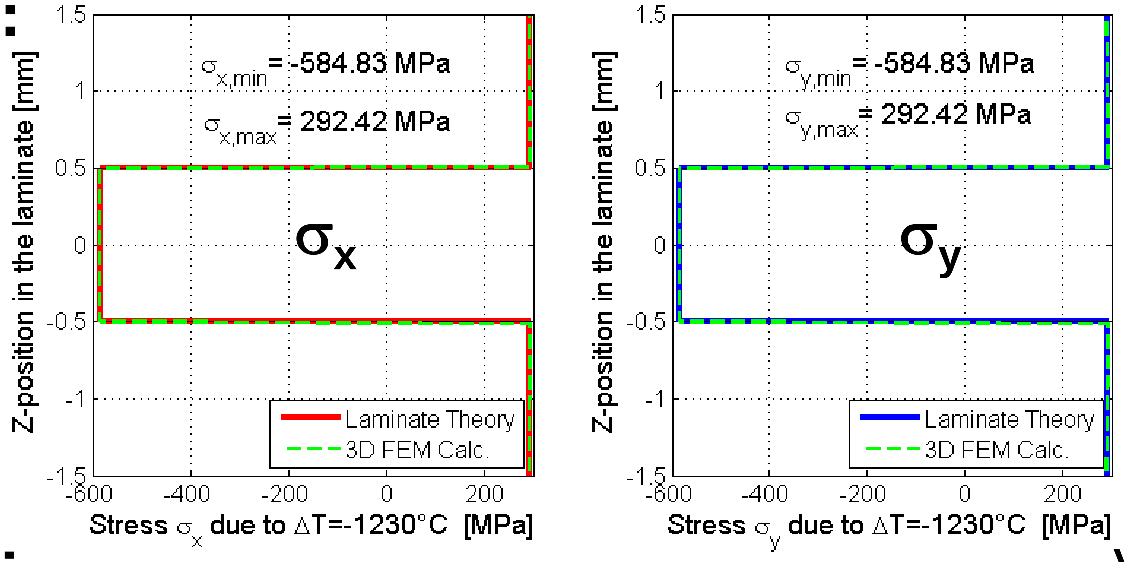
TH_{M2}=1mm

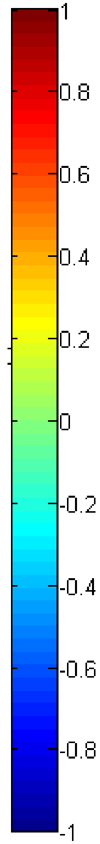
Property	**Units**	**M1**	**M2**
Young´s modulus *E*	MPa	390000	280000
Poissons ratio ν	-	0.22	0.22
CTE α (20-1200°C)	$10^{-6}K^{-1}$	9.8	8.0

Examples

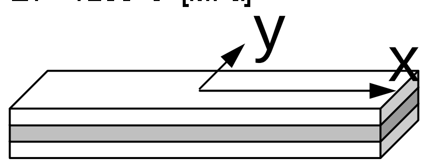
1) Symmetric laminate with isotropic layers:

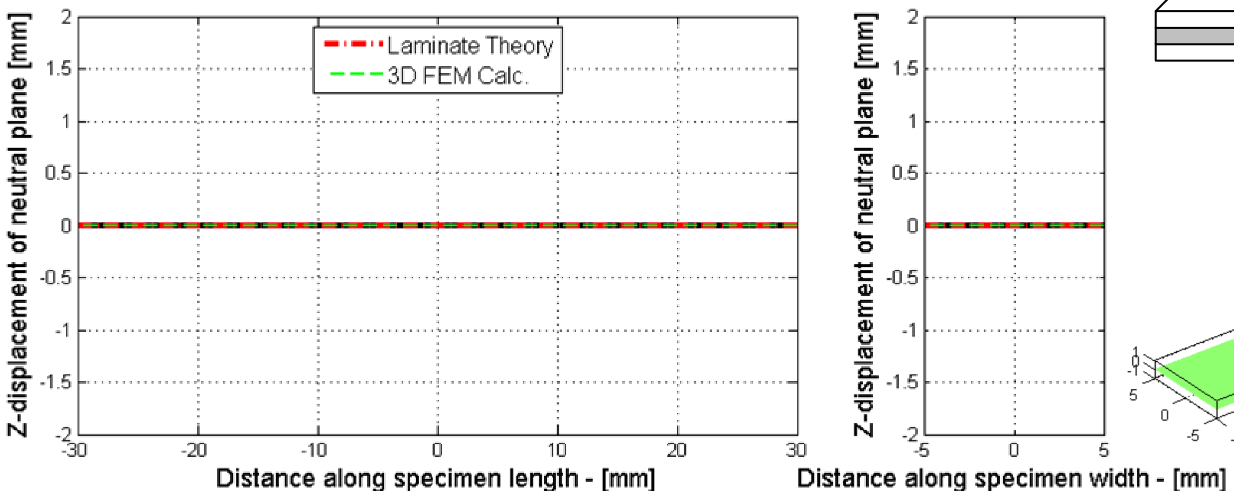
Stresses:

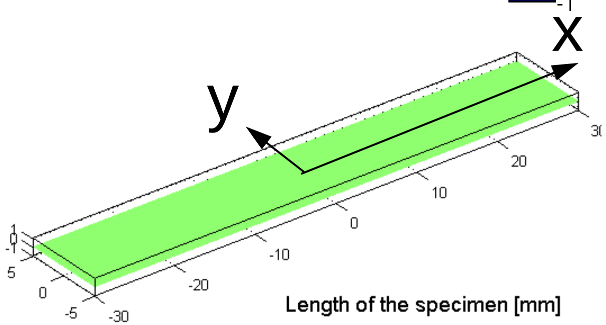




Deformation:

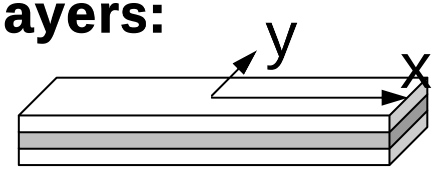




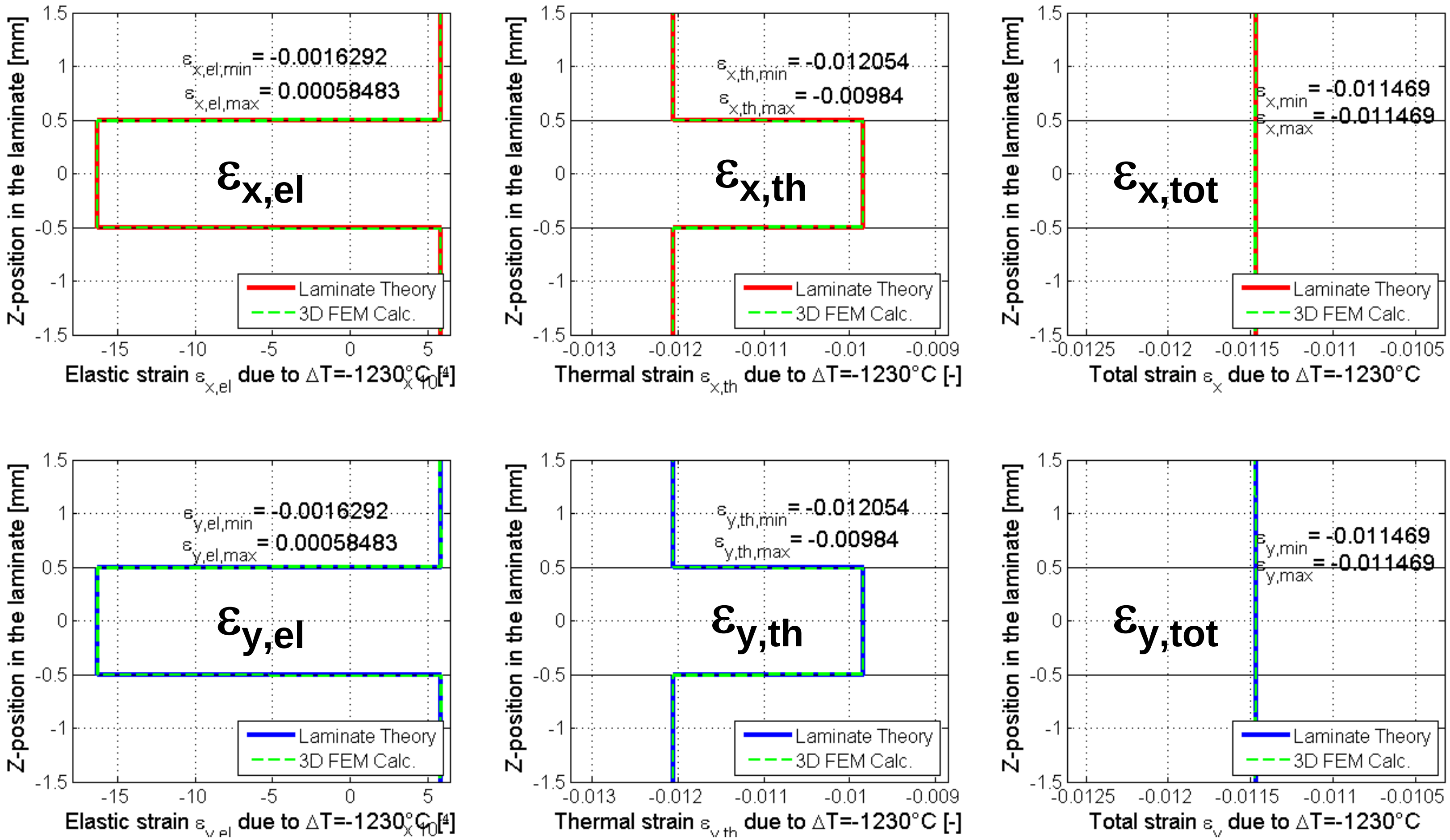


Examples

1) Symmetric laminate with isotropic layers:

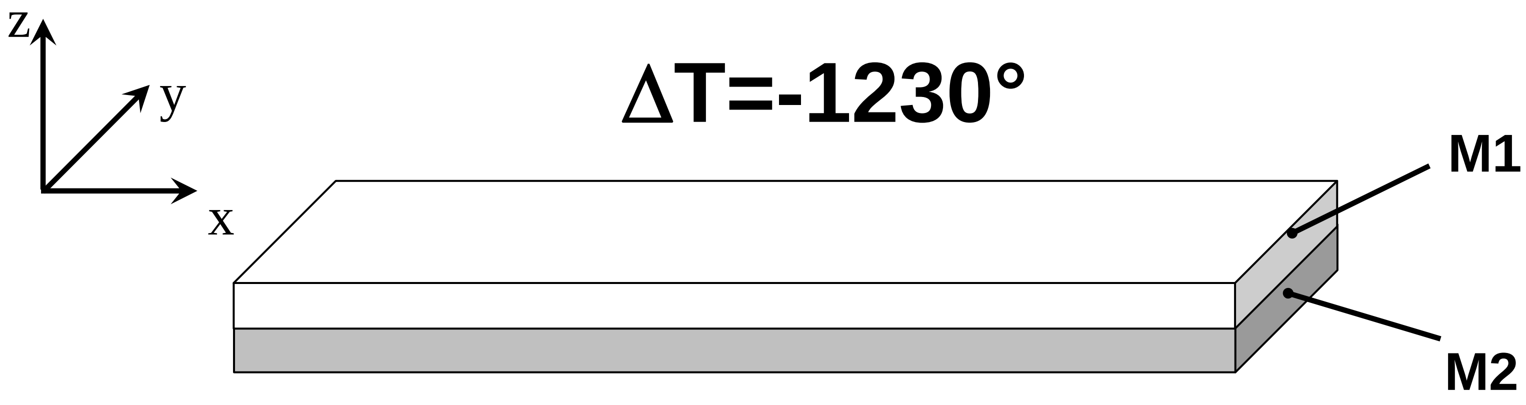


Strains:



Examples

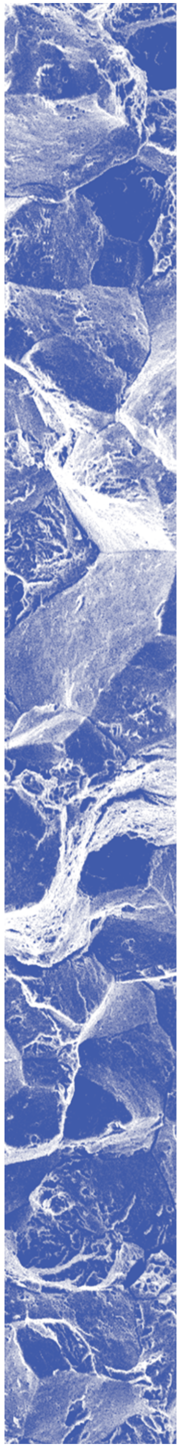
2) Non-symmetric laminate with isotropic layers:



TH_{M1}=1.5mm

TH_{M2}=1.5mm

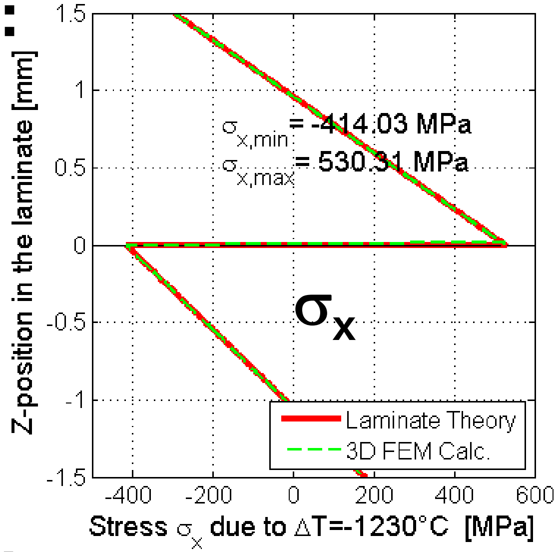
Property	Units	M1	M2
Young´s modulus E	MPa	390000	280000
Poissons ratio ν	-	0.22	0.22
CTE α (20-1200°C)	$10^{-6}K^{-1}$	9.8	8.0

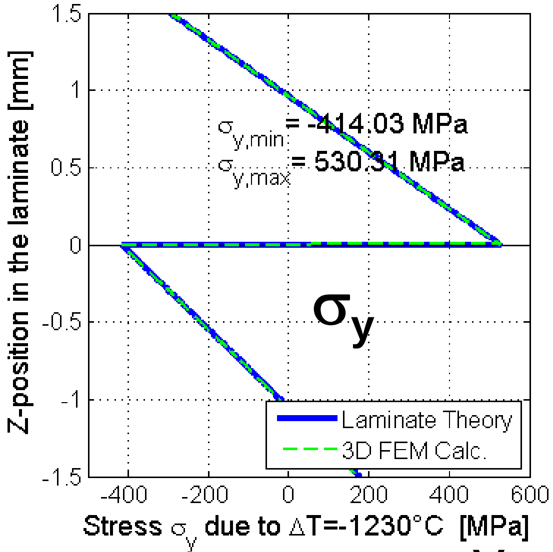


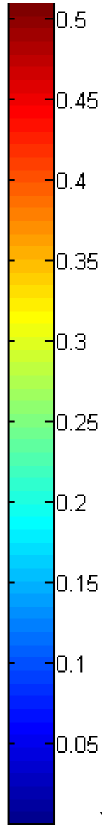
Examples

2) Non-symmetric laminate with isotropic layers:

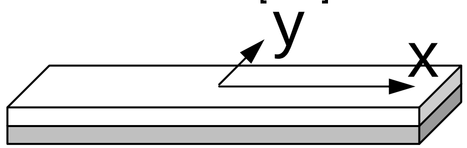
Stresses:

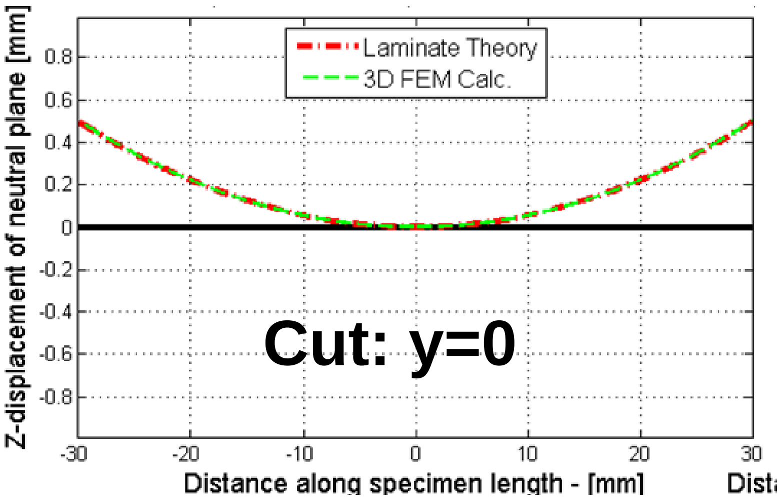


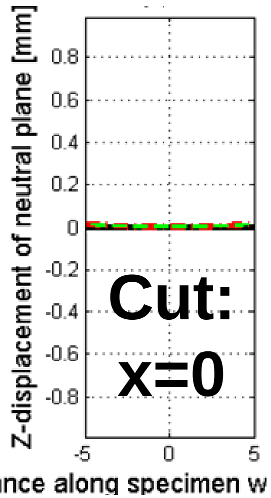


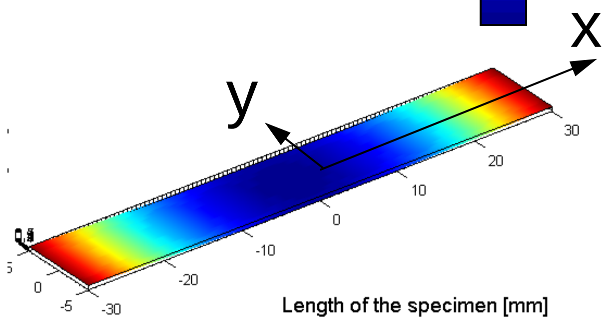


Deformation:



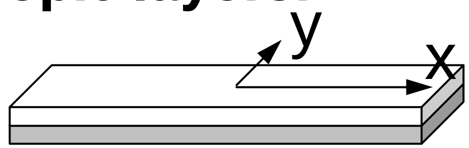




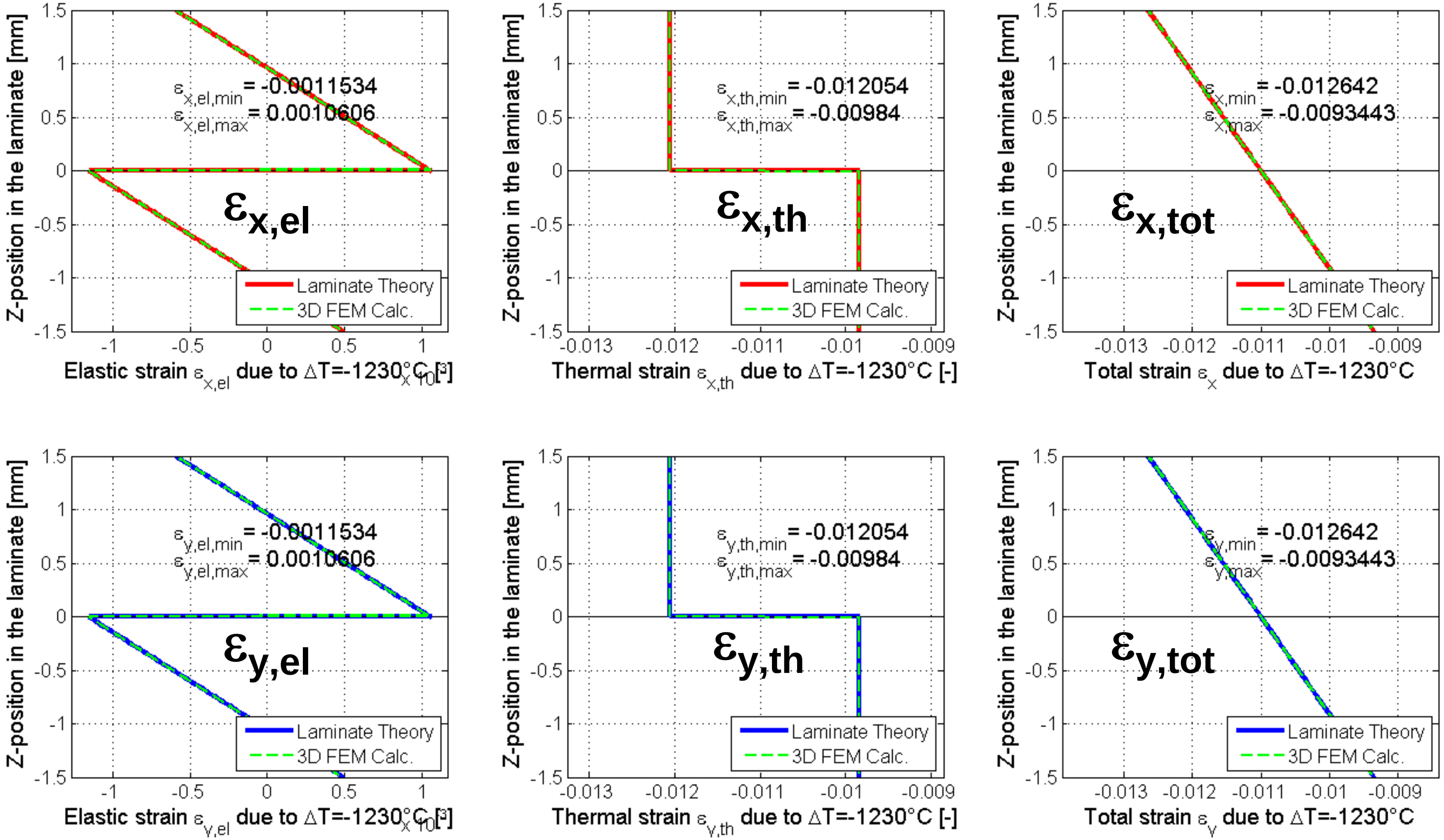


Examples

2) Non-symmetric laminate with isotropic layers:



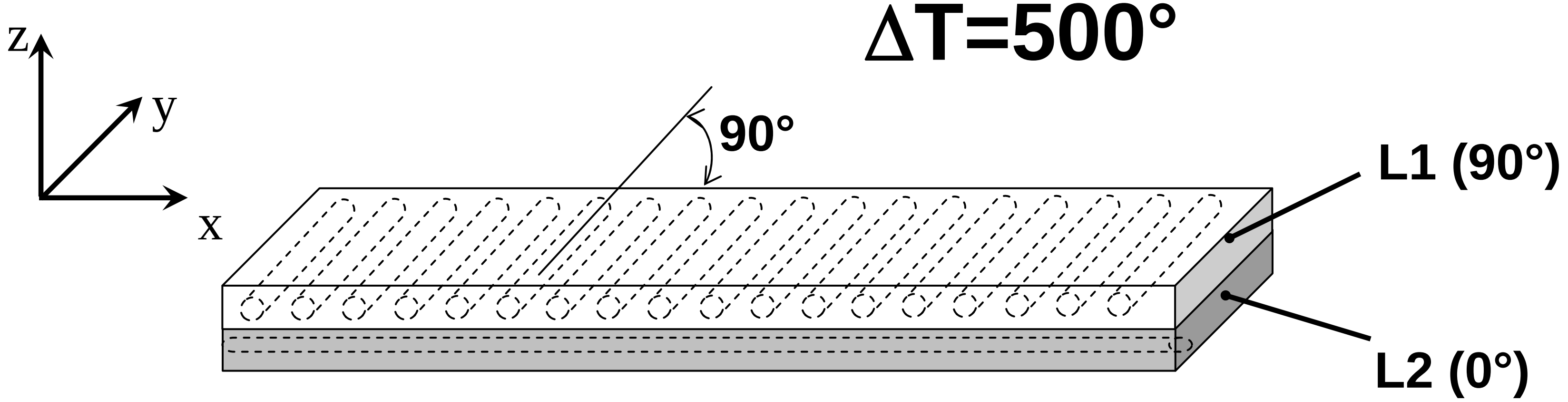
Strains



Examples

3a) Non-symmetric laminate with orthotropic layers:

$\Delta T=500°$

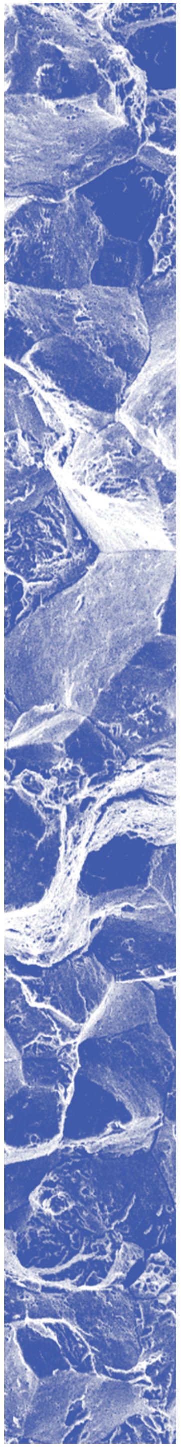


$TH_{M1}=1.5mm$ θ_{LX} (L1)=90°

$TH_{M2}=1.5mm$ θ_{LX} (L1)=0°

Property	Units	Value *
Young´s modulus E_L	MPa	76000
Young´s modulus E_T	MPa	5500
Poissons ratio $\nu_{LT(LZ)}$	-	0.238
Shear modulus $G_{LT(LZ)}$	MPa	2300
CTE α_L (20°C)	$10^{-6}K^{-1}$	-4
CTE α_T (20°C)	$10^{-6}K^{-1}$	80

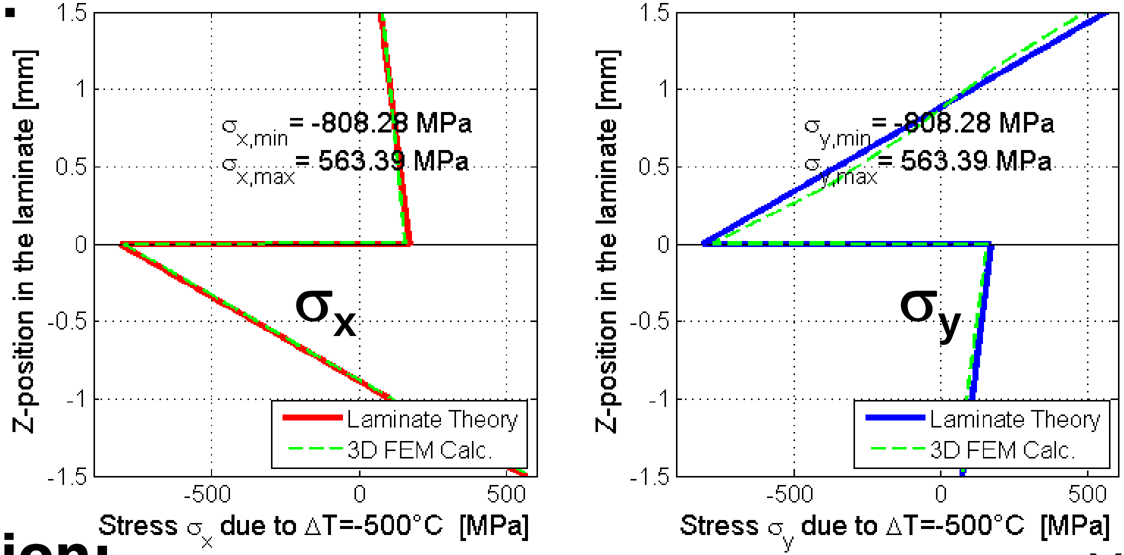
* Epoxy Matrix Composite reinforced by 50 % Kevlar fibers



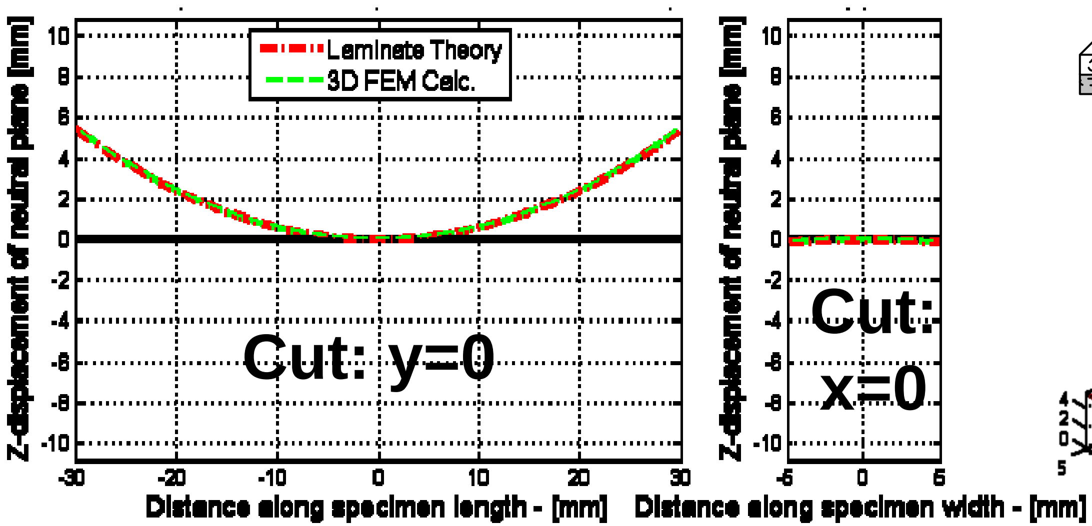
Examples

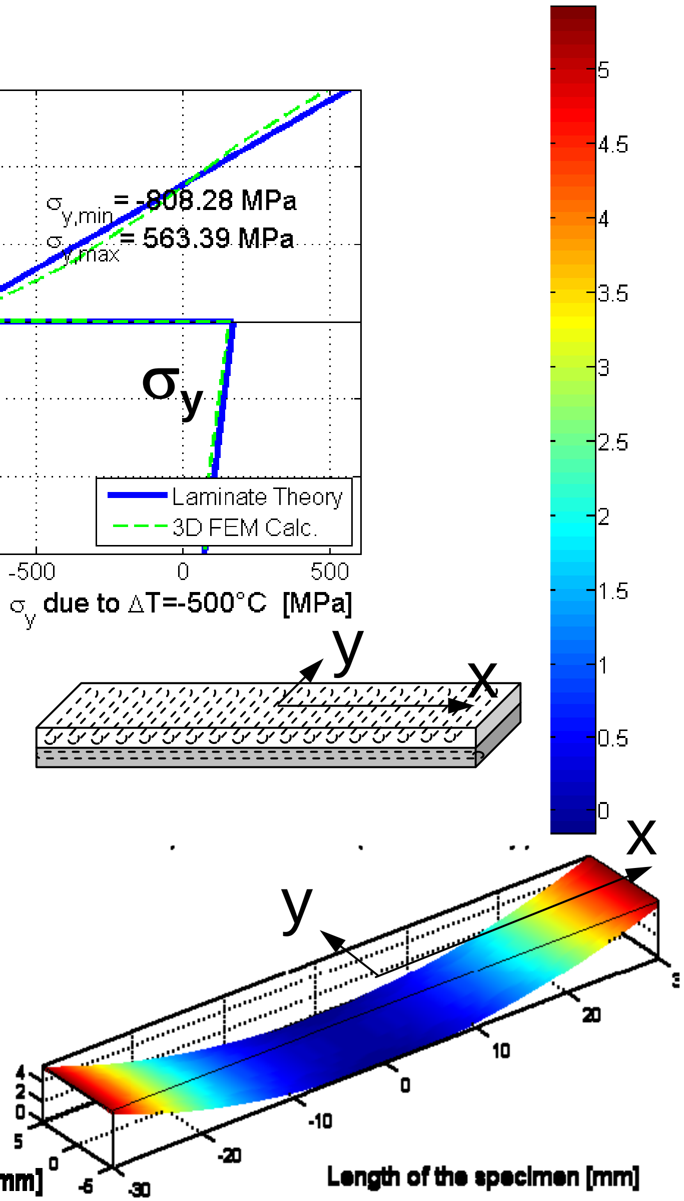
3a) Non-symmetric laminate with orthotropic layers:

Stresses:



Deformation:

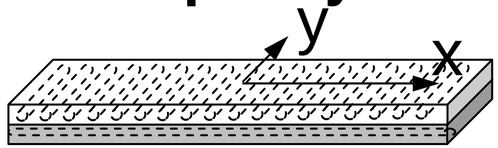


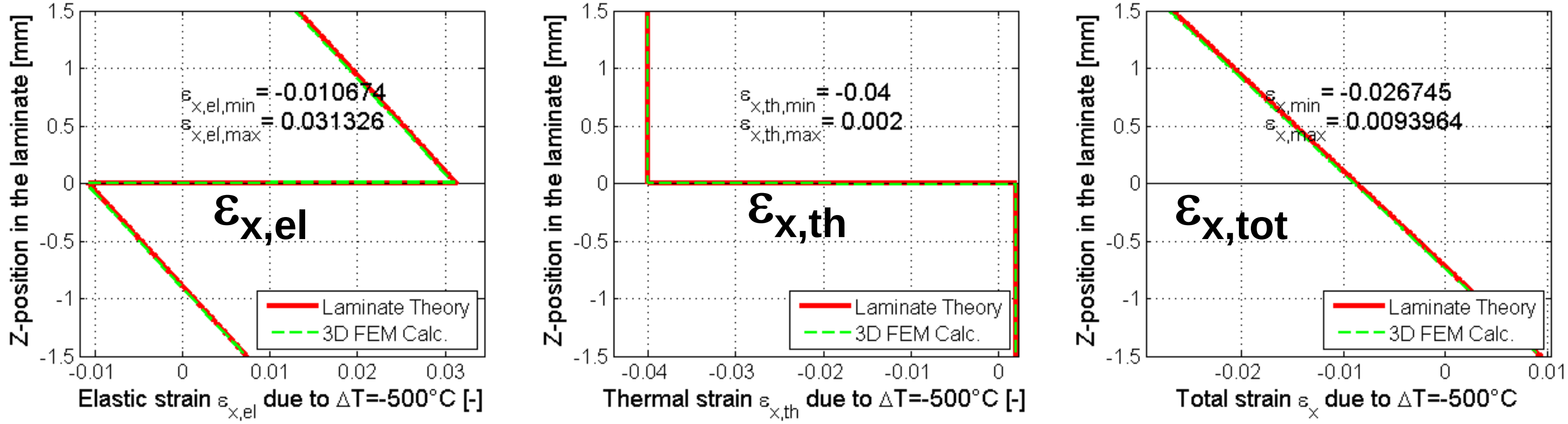


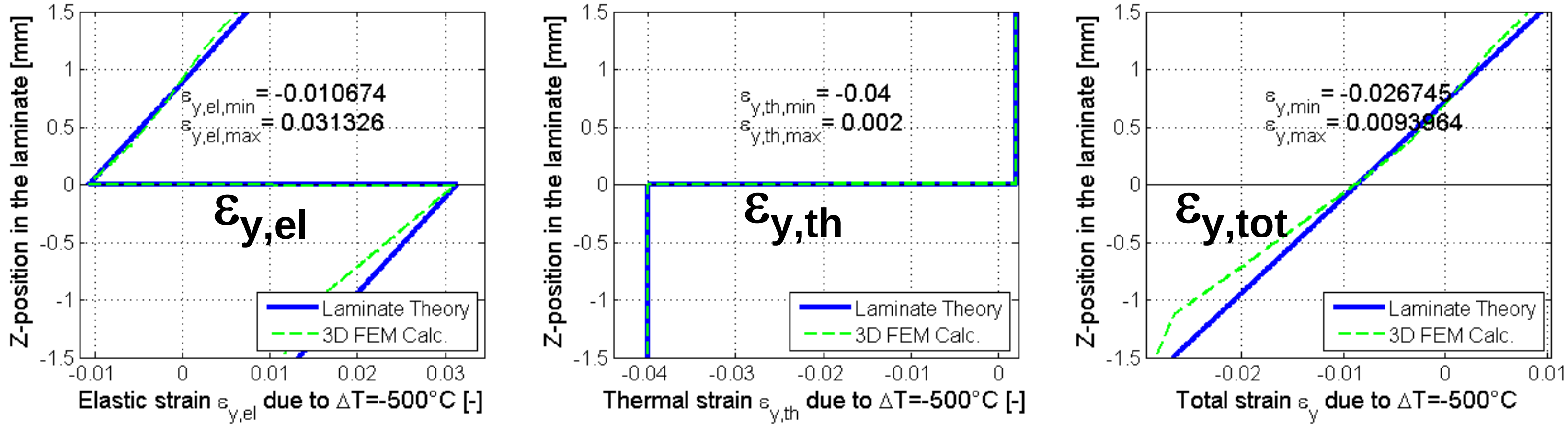
Examples

3a) Non-symmetric laminate with orthotropic layers:

Strains



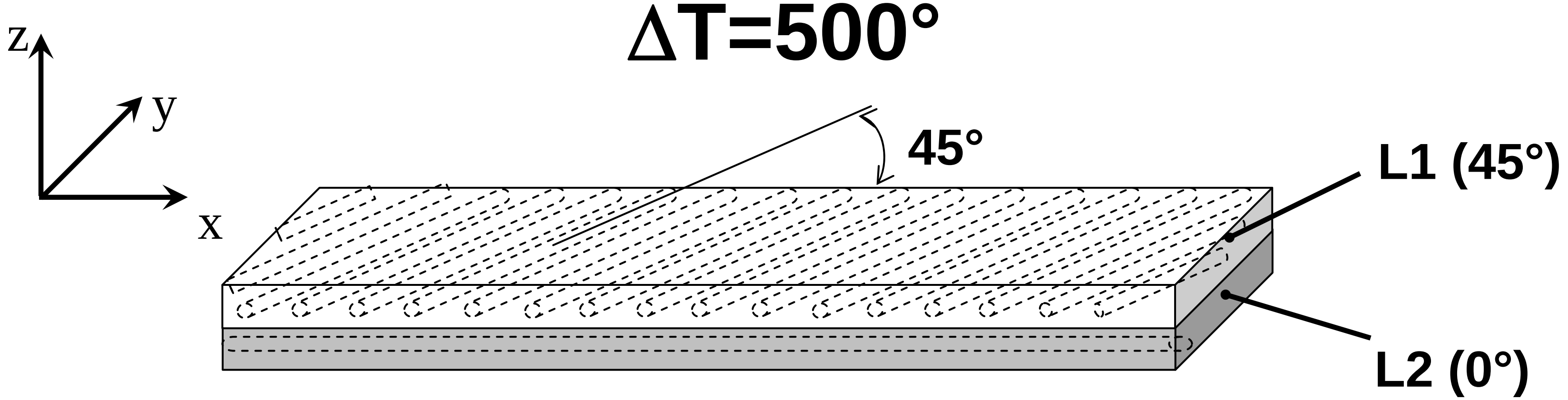




Examples

3b) Non-symmetric laminate with orthotropic layers:

$\Delta T=500°$

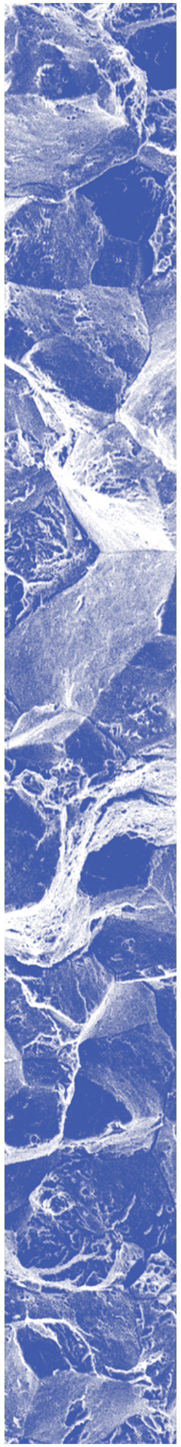


TH_{M1}=1.5mm θ_{LX} (L1)=45°

TH_{M2}=1.5mm θ_{LX} (L1)=0°

Property	Units	Value *
Young´s modulus E_L	MPa	76000
Young´s modulus E_T	MPa	5500
Poissons ratio $\nu_{ZL(TL)}$	-	0.238
Shear modulus $G_{ZL(TL)}$	MPa	2300
CTE α_L (20°C)	$10^{-6}K^{-1}$	-4
CTE α_T (20°C)	$10^{-6}K^{-1}$	80

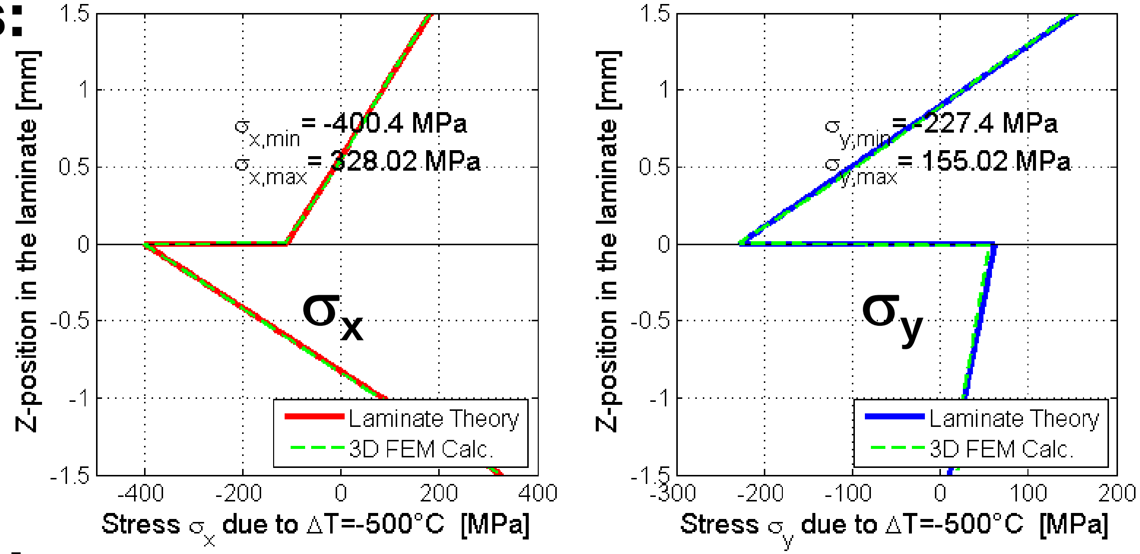
* **Epoxy Matrix Composite reinforced by 50 % Kevlar fibers**



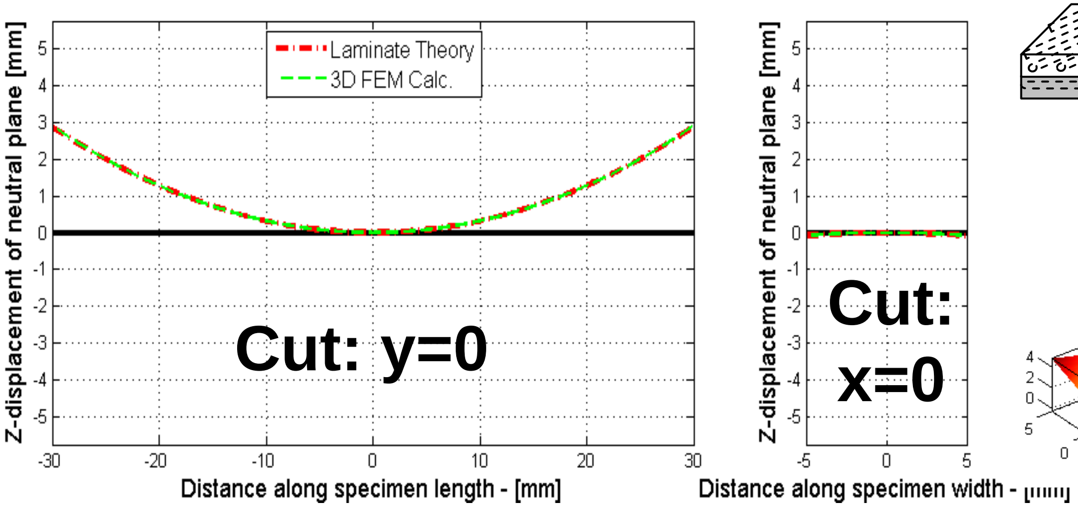
Examples

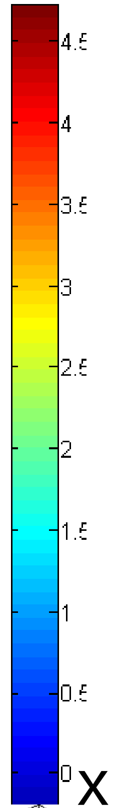
3b) Non-symmetric laminate with orthotropic layers:

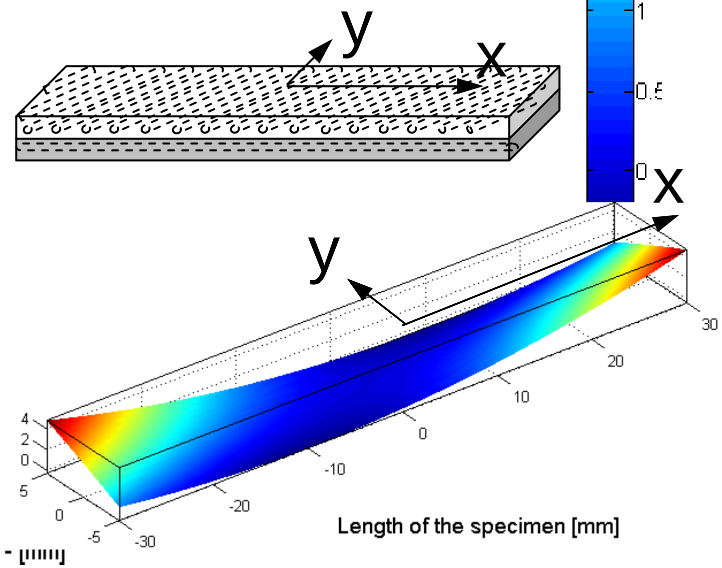
Stresses:



Deformation:



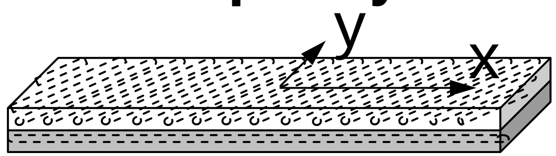


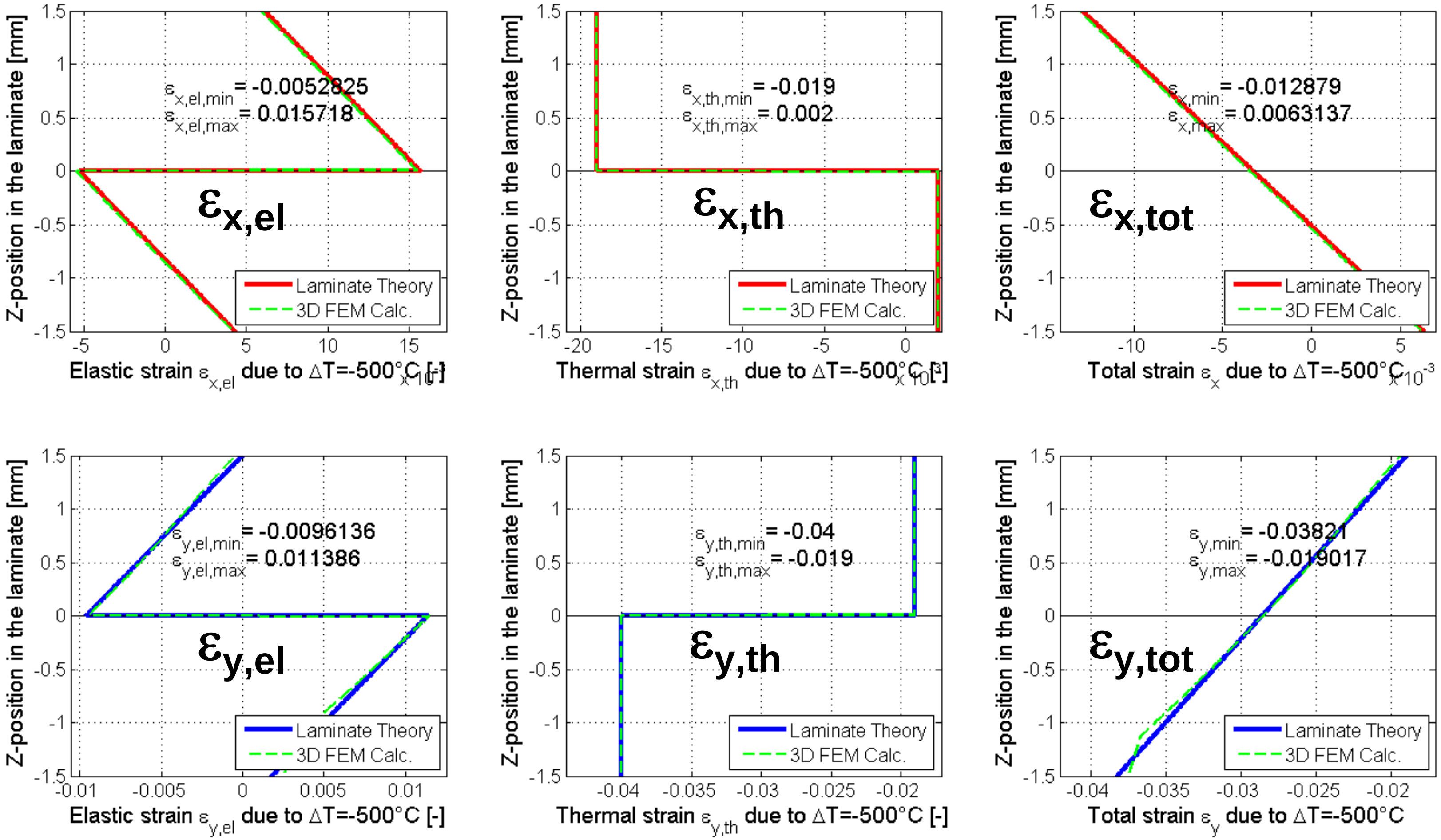


Examples

3b) Non-symmetric laminate with orthotropic layers:

Strains:



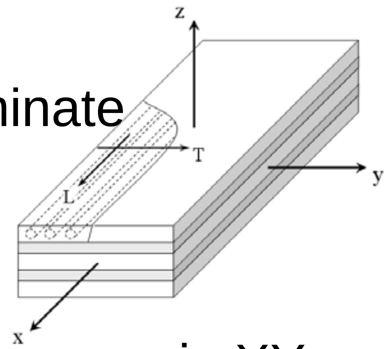


Summary

Model possibilities

CODE INPUT *):

- Number of layers + layer thicknesses.
- Material properties of each layer (isotropic or orthotropic with general fiber orientation).
- BC - External force, moment or ΔT applied on the laminate (also combination of them).



CODE OUTPUT *):

- Stress and strain (total, thermal) distribution over all layers – in XY or LT coordinate system.
- Deformation of the laminate midplane (bending and also twisting)
- Apparent material properties of the whole laminate (homogenization).
- Position of the neutral plane.
- And another... (e.g. ply failure analysis, ...).

*) Processed in Mathematica 8 or Matlab 2010

Conclusions

- Laminate theory and FE calculations are a in good agreement.
- Laminate theory is thus suitable for the fast laminate design (tailoring) – without need of FEM!

Application

- Design of the layer number, thicknesses or their material properties to:
  - reach some maximal (residual) stresses in each layer.
  - meet the requirements on the global laminate behaviour (total deformation).
- Determination of the critical laminate loading.
  - ...

Literature

- **Verbundwerkstoffe** – Vorlesungsbehelf zu den Vorlesungen, Inst. für Konstruieren in Kunst- und Verbundstoffen, MU Leoben.
- A.T.Nettles, **Basic Mechanics of Laminated Composite plates**, NASA Reference Publication, MSFC, Alabama, 1994.
- R.M. Jones, **Mechanics of Composite Materials** – 2. edition, Taylor & Francis, Philadelphia, 1999.